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A simple proof of the existence of adiabatic invariants for perturbed reversible problems

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Abstract
In this paper, we give a simple proof of the existence of invariants for reversible perturbations of action-angle systems. The originality of this proof is that it does not rely on canonical transformations that bring the system gradually closer to a normal form, but rather on a formal development of the invariant itself.

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1. Introduction

An adiabatic invariant is a property of a physical system which stays constant when changes are made slowly. In mechanics, an adiabatic change is a small perturbation of the Hamiltonian where the change of the energy is much slower than the orbital frequency (see, for instance, [Arn89, LM88]). The area enclosed by the different motions in phase space are then the adiabatic invariants. In the case of a perturbed Hamiltonian of the form

$$H(a,\theta) = H_0(a) + \varepsilon H_1(a,\theta), \tag{1.1}$$

with $(a,\theta) \in \mathbb{R} \times \mathbb{T}$, the classical procedure for deriving the invariants of motion is to look for a change of variables, close to the identity, in powers of ε,

$$\begin{aligned} I &= a + \varepsilon J_1(a,\theta) + \varepsilon^2 J_2(a,\theta) + \cdots \\ \varphi &= \theta + \varepsilon K_1(a,\theta) + \varepsilon^2 K_{2(a,\theta)} + \cdots \end{aligned} \tag{1.2}$$

in order to eliminate the angle variables of the Hamiltonian. This method, that goes back to Poincaré, was refined in the 20th century by Birkhoff [Bir27], Kolmogorov/Arnold/Moser (KAM) [Arn63, Kol54], Nekhoroshev [Nek77], and now forms the classical perturbation theory.

Using this coordinate transform method, the classical conclusion is that the series (1.2), though divergent, are asymptotic in the sense that, for instance,

$$|I(t) - a(t) - \varepsilon J_1(a(t), \theta(t)) - \cdots - \varepsilon^{k-1} J_{k-1}(a(t), \theta(t))| \leqslant C\varepsilon^k$$

for exponentially large time t. Hence, $I(t)$ is an adiabatic invariant for system (1.1), in the sense that its variation is small for a long-time interval.

In this paper, we consider perturbed *reversible* systems for which the classical method can be applied (see, for instance, [Mos73, Sev86, HLW06]). The systems we consider are of the following form:

$$\begin{aligned} \dot{a} &= \varepsilon s(a, \theta) \in \mathbb{R}^m, \\ \dot{\theta} &= \omega + \varepsilon \tau(a, \theta) \in \mathbb{T}^n, \end{aligned} \tag{1.3}$$

where ε is a small parameter, s is an odd function of θ and τ is an even function of θ,

$$\begin{aligned} s(a, -\theta) &= -s(a, \theta), \\ \tau(a, -\theta) &= \tau(a, \theta). \end{aligned} \tag{1.4}$$

For such systems, we propose an alternative construction of the invariants. It stems from the expansion of I itself and involves no change of variables in (a, θ): the procedure thus remains extremely basic. We assume here that ω is a constant vector, independent of a. This simplifies further some of the proofs while still covering most cases of interest[1]. We furthermore suppose that our model is nondegenerate, a not-so-serious limitation as most systems are nondegenerate (see [Arn89]).

Although the form of equations (1.4) seems very specific, a lot of systems in classical mechanics (reversible integrable ones to be precise) can be transformed into action-angle variables (see, for instance, [HLW06, chapter XI]). A prominent example of such a mechanical system is the Fermi–Pasta–Ulam model [FPU55] which nicely illustrates the persistence of adiabatic quantities (in this model, an adiabatic invariant is built up from the oscillatory energies of the stiff springs).

Results derived in this paper apply to the Fermi–Pasta–Ulam equations as much as to many other systems in celestial mechanics for instance. Moreover, they might be helpful to analyse geometric properties of numerical methods or to obtain stability results of a more theoretical nature such as those proved in [Moa02] or [HLW06, chapter XI].

2. The basic iterative scheme

Instead of studying coordinate transforms that bring (1.3) closer to some normal form, we search directly for an invariant of (1.3) of the form

$$I_\beta(a, \theta) = \beta \cdot a + \sum_{k \geqslant 1} \varepsilon^k J_k(a, \theta), \tag{2.1}$$

where $\beta \in \mathbb{R}^m$ and the functions J_k's are defined on $\mathbb{R}^m \times \mathbb{T}^n$. Here and in the following, the dot in $\beta \cdot a$ stands for the canonical scalar product of vectors β and a. In order to obtain a formula for the J_k's, we compute the (formal at this stage) derivative along the exact solution of (1.3),

[1] The case of varying frequencies is more technically intricate and would require ultraviolet cut-off techniques. It is out of the scope of this paper.

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\beta(a,\theta) = \beta \cdot \dot{a} + \sum_{k\geqslant 1} \varepsilon^k ((\partial_a J_k)\dot{a} + (\partial_\theta J_k)\dot{\theta}),$$
$$= \varepsilon\beta \cdot s + \sum_{k\geqslant 1} (\varepsilon^{k+1} s \cdot (\partial_a J_k) + \varepsilon^k \omega \cdot (\partial_\theta J_k) + \varepsilon^{k+1} \tau \cdot (\partial_\theta J_k)),$$
$$= \sum_{k\geqslant 1} \varepsilon^k (G_k + \omega \cdot (\partial_\theta J_k)), \tag{2.2}$$

where

$$G_1(a,\theta) = \beta \cdot s(a,\theta)$$

and

$$G_k(a,\theta) = s(a,\theta) \cdot (\partial_a J_{k-1})(a,\theta) + \tau(a,\theta) \cdot (\partial_\theta J_{k-1})(a,\theta) \tag{2.3}$$

for $k \geqslant 2$. Hence, the function $I_\beta(a,\theta)$ is an invariant of (1.3) if the functions J_k satisfy

$$\forall k \geqslant 1, \qquad \omega \cdot (\partial_\theta J_k)(a,\theta) + G_k(a,\theta) = 0. \tag{2.4}$$

For $k = 1$, this equation yields

$$\forall (a,\theta) \in \mathbb{R}^m \times \mathbb{T}^n, \qquad \beta \cdot s(a,\theta) + \omega \cdot (\partial_\theta J_1)(a,\theta) = 0. \tag{2.5}$$

Since J_1 is required to be 2π-periodic, the average over the torus $\mathbb{T}^n$ of $\beta \cdot s(a,\theta)$ must vanish, i.e.,

$$\int_{\mathbb{T}^n} \beta \cdot s(a,\theta)\,\mathrm{d}\theta = 0. \tag{2.6}$$

Equation (2.5) then becomes solvable, as stated by lemma X.4.1 of [HLW06]. It is important at this stage to underline the fundamental role of the reversibility assumption (1.4). As a matter of fact, this condition ensures that the integral (2.6) is null, as can be seen from the elementary calculus

$$\int_{\mathbb{T}^n} \beta \cdot s(a,\theta)\,\mathrm{d}\theta = \int_{\mathbb{T}^n_+} \beta \cdot s(a,\theta)\,\mathrm{d}\theta + \int_{\mathbb{T}^n_-} \beta \cdot s(a,\theta)\,\mathrm{d}\theta$$
$$= \int_{\mathbb{T}^n_+} \beta \cdot s(a,\theta)\,\mathrm{d}\theta - \int_{\mathbb{T}^n_+} \beta \cdot s(a,\theta)\,\mathrm{d}\theta = 0,$$

where we have assumed, for instance, that $\mathbb{T}^n = [-\pi;\pi]^n$ so that $\mathbb{T}^n = \mathbb{T}^n_+ \cup \mathbb{T}^n_-$ with $\mathbb{T}_+ = [0,\pi] \times \mathbb{T}^{n-1}$ and $\mathbb{T}_+ = [-\pi;0] \times \mathbb{T}^{n-1}$. At each step, one needs the solution of equation (2.4) to be even with respect to θ: assume that J_{k-1} is known and even. Taking into account that

- s is odd w.r.t. θ,
- τ is even w.r.t. θ,
- $\partial_a J_{k-1}$ is even and $\partial_\theta J_{k-1}$ is odd ,

we see that G_k in (2.3) is odd and hence of zero average, so that J_k exists and is even.

3. Main result

Our construction requires a slight refinement of lemma X.4.1 of [HLW06], which we now formulate together with some estimates using the following norms: let

$$U_\rho = \{\theta \in \mathbb{T}^n + \mathrm{i}\mathbb{R}^n;\ \|\Im(\theta)\| \leqslant \rho\},$$

where $\|\cdot\|$ denotes the maximum norm in $\mathbb{R}^n$. If F is a real-analytic function from $B_r(a_0) \times U_\rho$ onto $\mathbb{C}$, where for $r > 0$, $B_r(a_0)$ is the complex ball of radius r and centre $a_0 \in \mathbb{R}^m$, we denote

$$\|F\|_{r,\rho} = \sup_{(a,\theta)\in B_r(a_0)\times U_\rho} |F(a,\theta)|,$$

and whenever F is vector valued, say $F \in \mathbb{C}^m$,

$$\|F\|_{r,\rho} = \sum_{i=1}^{m} \|F_i\|_{r,\rho}.$$

Lemma 3.1. *Suppose $\omega \in \mathbb{R}^n$ satisfies the diophantine condition*

$$\exists\gamma > 0, \qquad \exists\nu > 0, \qquad \forall\alpha \in \mathbb{Z}^n/\{0\}, \qquad |\alpha\cdot\omega| \geqslant \gamma|\alpha|^{-\nu}. \tag{3.1}$$

Let $a_0 \in \mathbb{R}^m$, and consider positive numbers r and ρ and let G be an analytic function on $B_r(a_0) \times U_\rho$. Let $\langle G\rangle$ denote the average of G over $\mathbb{T}^n$. Then, for all positive $\delta < \min(1,\rho)$ and $\rho < r$, the equation

$$\omega\cdot\partial_\theta J + G = \langle G\rangle \tag{3.2}$$

has a unique analytic solution J on $B_r(a_0) \times U_{\rho-\delta}$ with zero average $\langle J\rangle = 0$ on $\mathbb{T}^n$, and we have the estimates

$$\|J\|_{r,\rho-\delta} \leqslant \kappa_0\delta^{-\eta+1}\|G\|_{r,\rho} \qquad \text{and} \qquad \|\partial_\theta F\|_{r,\rho-\delta} \leqslant \kappa_1\delta^{-\eta}\|G\|_{r,\rho}, \tag{3.3}$$

where $\eta = \nu+n+1$, $\kappa_0 = \gamma^{-1}8^n2^{\nu+1}\nu!$ and $\kappa_1 = \gamma^{-1}8^n2^{\nu+2}(\nu+1)!$. Moreover, if G is an odd function of θ, J is an even one.

Proof. We take over the proof of lemma 3.1 in order to show that J is even as soon as G is odd: denoting

$$G(a,\theta) = \sum_{\alpha\in\mathbb{Z}^n} g_\alpha(a)\,\mathrm{e}^{\mathrm{i}\alpha\cdot\theta} \qquad \text{and} \qquad J(a,\theta) = \sum_{\alpha\in\mathbb{Z}^n} j_\alpha(a)\,\mathrm{e}^{\mathrm{i}\alpha\cdot\theta}$$

the Fourier expansions of G and J, we have for a nonzero $\alpha \in \mathbb{Z}^n$, $j_\alpha(a) = -\frac{g_\alpha(a)}{i\alpha\cdot\omega}$. The function G being odd, $g_\alpha(a) = -g_{-\alpha}(a)$ for all $\alpha \in \mathbb{Z}^n$ and all $a \in B_r(a_0)$, so that

$$j_{-\alpha}(a) = -\frac{g_{-\alpha}(a)}{i(-\alpha)\cdot\omega} = -\frac{g_\alpha(a)}{i\alpha\cdot\omega} = j_\alpha(a),$$

i.e. J is even. It has zero average since $g_0 = 0$. The estimates (3.3) are then obtained just as in lemma X.4.1 of [HLW06]. □

We are now in position to state the main result of this paper.

Theorem 3.2. *Assume that the functions s and τ are analytic on $B_r(a_0) \times U_\rho$ for a given $a_0 \in \mathbb{R}^m$ and for given numbers $r > 0$ and $\rho > 0$, and satisfy conditions (1.4), i.e. that s is odd and ρ even w.r.t. θ. Suppose in addition that the vector ω is constant and satisfies condition (3.1). Then, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ and for any $\beta \in \mathbb{R}^m$, there exists a function $I_\beta(a,\theta)$ analytic on $B_{r/2}(a_0) \times U_{\rho/2}$ and such that*

$$\|I_\beta(a,\theta) - \beta\cdot a\|_{r/2,\rho/2} \leqslant C_0\varepsilon \tag{3.4}$$

for some constant C_0 depending on bounds of the derivatives of s and τ. Moreover, if $(a(t),\theta(t))$ denotes a solution of (1.3) starting at $a(0) \in B_{r/2}(a_0)$, then as long as $a(t) \in B_{r/2}(a_0)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\beta(a(t),\theta(t)) = R(a(t),\theta(t)), \tag{3.5}$$

where $R(a,\theta)$ satisfies

$$\|R\|_{r/2,\rho/2} \leqslant \exp\left(-\frac{c_0}{\varepsilon^\sigma}\right) \tag{3.6}$$

with $\sigma = 1/\eta$, $\eta = \nu + n + 1$ and where c_0 is a constant depending on r, ρ, γ, ν, n and the functions s and τ.

Proof. As soon as $k \geqslant 2$, the ε^k-term in the derivative of I_β (compare (2.2)) vanishes for all (a,θ) if and only if

$$\omega \cdot (\partial_\theta J_k) = -s \cdot (\partial_a J_{k-1}) - \tau \dot{(}\partial_\theta J_{k-1}),$$

where, for brevity, we have omitted the arguments (a,θ) in the functions s, τ and J_k. We now proceed by induction. We have already established that J_1 is even. Assume that there exist real-analytic functions $J_1, \ldots, J_{k-1}$, $k \geqslant 2$, 2π-periodic and even w.r.t. θ. Then, the function $(\partial_a J_{k-1})$ is even w.r.t. θ and the function $(\partial_\theta J_{k-1})$ is odd, so that, upon applying the rule

$$\text{odd} \times \text{even} = \text{odd},$$

we deduce that the function $s \cdot (\partial_a J_{k-1}) + \tau \cdot (\partial_\theta J_{k-1})$ is odd and hence of zero average. The hypotheses of lemma 3.1 hold with $G = s \cdot (\partial_a J_{k-1}) + \tau \cdots (\partial_\theta J_{k-1})$ and $\langle G \rangle = 0$, thus ensuring the existence of a real-analytic function J_k which is even w.r.t. to θ.

Assume that J_{k-1} is analytic on $B_{r_*}(a_0) \times U_{\rho_*}$ for some positive numbers $r_* < r$ and $\rho_* < \rho$. Let $\mu < \min(1, r_*)$ and $\delta < \min(1, \rho_*)$ be positive constants. We have, using (3.3),

$$\|J_k\|_{r_*-\mu,\rho_*-\delta} \leqslant 2^{\eta-1}\kappa_0 M \delta^{-\eta+1}(\|\partial_\theta J_{k-1}\|_{r_*-\mu,\rho_*-\delta/2} + \|\partial_a J_{k-1}\|_{r_*-\mu,\rho_*-\delta/2}).$$

Now, as J_{k-1} is analytic on $B_{r_*}(a_0) \times U_{\rho_*}$, we obtain using Cauchy estimates

$$\|\partial_\theta J_{k-1}\|_{r_*-\mu,\rho_*-\delta/2} \leqslant \frac{2}{\delta}\|J_{k-1}\|_{r_*-\mu,\rho_*}$$

and

$$\|\partial_a J_{k-1}\|_{r_*-\mu,\rho_*-\delta/2} \leqslant \frac{1}{\mu}\|J_{k-1}\|_{r_*,\rho_*-\delta/2}.$$

Gathering previous estimates, we thus get

$$\|J_k\|_{r_*-\mu,\rho_*-\delta} \leqslant C\delta^{-\eta+1}\left(\frac{1}{\delta} + \frac{1}{\mu}\right)\|J_{k-1}\|_{r_*,\rho_*},$$

where C is a constant independent of δ and μ. For $L \in \mathbb{N}$, let $\delta = \rho/(2L)$ and $\mu = r/(2L)$. By induction we easily obtain

$$\|J_k\|_{r/2,\rho/2} \leqslant C(cL)^{\eta L}$$

for some constants C and c depending on n, ν, γ and M. We then define

$$I_\beta(a,\theta) = \beta \cdot a + \sum_{k=1}^{K} \varepsilon^k J_k(a,\theta)$$

with the optimal truncation index $K = \text{Floor}((e\varepsilon c)^{-1/\eta})$ (see, for instance, [HLW06, Nek77]). □

Corollary 3.3. *Under the hypotheses of theorem 3.2, ε_0 can be taken sufficiently small so that the following holds: let $(a(t),\theta(t))$ be a solution of (1.3) such that $a(0) \in B_{r/4}(a_0)$. Then we have for all $t \leqslant \exp(c_0\varepsilon^{-\sigma}/2)$,*

$$|I_\beta(a(t),\theta(t)) - I_\beta(a(0),\theta(0))| \leqslant \exp(-c_0\varepsilon^{-\sigma}/2), \tag{3.7}$$

where c_0 is the constant appearing in (3.6) and

$$|\beta \cdot a(t) - \beta \cdot a(0)| \leqslant C_\beta \varepsilon$$

for some constant C_β independent of ε.

Proof. It is clear that (3.7) is valid as long as we have $a(t) \in B_{r/2}(a_0)$. This result combined with (3.4) for β scanning all vectors of the canonical basis of $\mathbb{R}^m$ leads to

$$\|a(t) - a(0)\| \leqslant C_0 \varepsilon$$

for some constant C_0, as long as $a(t) \in B_{r/2}(a_0)$. Hence, we deduce that if $\varepsilon_0 \leqslant r/(4C_0)$, we have $a(t) \in B_{r/2}(a_0)$ for $t \leqslant \exp(c_0\varepsilon^{-\sigma}/2)$ which completes the proof. □

4. Some comments on the resonant case

Suppose that the diophantine condition (3.1) holds only for α's such that $\alpha \cdot \omega \neq 0$, a set which is not assumed to be reduced to $\{0\} \subset \mathbb{Z}^n$, in contrast with the situation considered before:

Definition 4.1. *For a given set of frequencies $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$, the resonance module $\mathcal{M}$ is defined as*

$$\mathcal{M} = \{\alpha \in \mathbb{Z}^n | \alpha_1\omega_1 + \cdots + \alpha_n\omega_n = 0\}.$$

The vector of frequencies ω is said to be non-resonant outside $\mathcal{M}$ if

$$\exists \gamma, \nu > 0, \qquad \forall \alpha \in \mathbb{Z}^n \backslash \mathcal{M}, \qquad |\alpha \cdot \omega| > \gamma |\alpha|^{-\nu}. \tag{4.1}$$

The orthogonal of the resonant module is defined by

$$\mathcal{M}^\perp = \{\beta \in \mathbb{Z}^n | \forall \alpha \in \mathcal{M}, \alpha_1\beta_1 + \cdots + \alpha_n\beta_n = 0\}.$$

Under the assumptions (1.4), equation (1.3) admits a formal invariant of the form (2.1) if and only if equations (2.4) hold for all $k \geqslant 1$. At each step k, we thus have to solve once again the homological equation

$$\omega \cdot \partial_\theta J + G = 0. \tag{4.2}$$

Now, in contrast with the nondegenerate case, $\langle G \rangle = 0$ is not a sufficient condition to ensure the existence of a solution. As a matter of fact,

$$\omega \cdot \partial_\theta J(a, \theta) = \sum_{\alpha \in \mathbb{Z}^n} (\omega \cdot \alpha) j_\alpha(a)\, \mathrm{e}^{\mathrm{i}\alpha\cdot\theta} = \sum_{\alpha \in \mathbb{Z}^n/\mathcal{M}} (\omega \cdot \alpha) j_\alpha(a)\, \mathrm{e}^{\mathrm{i}\alpha\cdot\theta},$$

so that one should have

$$\omega \cdot g_\alpha = 0 \qquad \text{for all} \quad \alpha \in \mathcal{M}, \tag{4.3}$$

a condition which is not satisfied in general. Consider for instance the system

$$\begin{cases} \dot{a} = \varepsilon \sin(\theta_1 - \theta_2) \\ \dot{\theta}_1 = 1 + \varepsilon \\ \dot{\theta}_2 = 1 + 2\varepsilon, \end{cases}$$

with exact solution

$$\begin{cases} a(t) = a(0) + \cos(\varepsilon t) \\ \theta_1(t) = \theta_1(0) + (1 + \varepsilon)t \\ \theta_2(t) = \theta_2(0) + (1 + 2\varepsilon)t. \end{cases}$$

We see that $a(t)$ is not an adiabatic invariant, implying that (4.3) is indeed necessary. Even when condition (4.3) is fulfilled, the construction cannot be carried on further than $k = 2$. It seems that in this situation, a more elaborate analysis is needed.

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